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Hausdorff Dimension of Sets of Generic Points for Gibbs Measures

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For a translation invariant Gibbs measure ν on the configuration space X of a lattice finite spin system, we consider the set X_ν of generic points. Using a Breiman type convergence theorem on the set X_μ of generic points of an arbitrary translation invariant probability measure μ on X, we evaluate the Hausdorff dimension of the set X_ν with respect to any metric out of a wide class of "scale" metrics on X (including Billingsley metrics generated by Gibbs measures).

KEY WORDS: Hausdorff dimension; Gibbs measures; generic points.

1. INTRODUCTION

Let (X, λ) be a compact metric space, τ an action of the group $\mathbb{Z}^d$ or one of its subsemigroups by continuous transformations of X, and μ a τ-invariant Borel probability measure on X. A point $x \in X$ is called *generic* with respect to μ or, shortly, *μ-generic* if the averages of each continuous function $f: X \to \mathbb{R}$ over increasing pieces of the τ-orbit of x converge to the mean value $\int f\, d\mu$. It is the generic points that provide the stability of time averages, consistency of statistical estimators, and the existence of specific characteristics describing the "macro" properties of systems in statistical physics. It is clear that one may be interested in evaluation of the "size" of the set X_μ consisting of all μ-generic points.

According to the individual ergodic theorem, $\mu(X_\mu) = 1$ if μ is ergodic, i.e., in this case, X_μ is "comparable" with the whole space X. On the other

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hand, if μ is non-ergodic, $\mu(X_\mu)=0$.[3] In the latter case X_μ is "comparable" with the empty set. Although a statistician may be satisfied with this rather restricted information, a more refined description of this set is of significant interest. Since X is a metric space it is natural to evaluate some metric characteristics of X_μ, in particular, its Hausdorff dimension.

In the case where X is a sequence space, τ consists of translations, and λ is the Billingsley metric (see Section 2) specified by a τ-invariant Markov measure ν, an impressive study of the Hausdorff dimension $D_\lambda(X_\mu)$ of X_μ (and some more general sets) was performed by Cajar.[1] He proved that $D_\lambda(X_\mu)=h(\mu)/h(\nu,\mu)$, where $h(\mu)$ is the entropy of μ and $h(\nu,\mu)$ is an entropy type joint characteristic of μ and ν. More recently Olivier[9] has extended this result to Billingsley metrics specified by invariant g-measures.

In this paper we assume that S is a finite set, T is a subsemigroup of $\mathbb{Z}^d$ (one can think of T as $\mathbb{Z}^d$ or $\mathbb{Z}^d_+$), $X^{(T)}=S^T=\{x\colon T\to S\}$ is the space of S-valued configurations on T endowed with a "scale metric" defined in ref. 15 (see Section 3 below), and $\tau=\tau^{(T)}$ is the semigroup of all translations on $X^{(T)}$. We study the Hausdorff dimension of the set $X^{(T)}_\mu$ of μ-generic points, where μ is the natural projection to $X^{(T)}$ of a Gibbs measure defined on $X^{(\mathbb{Z}^d)}$.

To be more specific we start with the following definition. Let $\mathscr{T}=\{T_n\}$ be a sequence of finite subsets of T and μ a Borel probability measure on X. We say that a point $x\in X^{(T)}$ is *μ-generic with respect to $\mathscr{T}$* if for any function $f\in C(X^{(T)})$ we have

$$\lim_{n\to\infty} |T_n|^{-1}\sum_{t\in T_n} f(\tau_t x)=\int f\,d\mu \tag{1.1}$$

(we denote by $|A|$ the cardinality of a set $A\subset\mathbb{Z}^d$). This definition determines the set $X^{(T)}_\mu=X^{(T)}_\mu(\mathscr{T})$ of all μ-generic points (with respect to $\mathscr{T}$) studied in the paper.

We study the Hausdorff dimension of the set $X^{(T)}_\mu$ with respect to metrics out of a wide class containing Billingsley metrics specified by Gibbs measures. For a Gibbs measure μ we prove that $D_\lambda(X^{(T)}_\mu)=h(\mu)/\kappa$ where $h(\mu)$ is the entropy of μ and κ is a parameter specified by the metric λ.

We want to stress that μ is not assumed to be ergodic. Our result implies that for a given metric λ the dimension $D_\lambda(X^{(T)}_\mu)$ is the same for all Gibbs measures μ with the same entropy, whereas $\mu(X_\mu)=1$ if μ is ergodic and $\mu(X_\mu)=0$ if μ is not ergodic. For example, this is the case for the two-dimensional Ising ferromagnet at zero external field: in this model for a

[3] Indeed, if $\{\mu_\alpha\}$ is the ergodic decomposition of μ, for each α there is a continuous function f_α with $\int f_\alpha\,d\mu_\alpha\neq\int f_\alpha\,d\mu$; hence $X_\mu\cap\bigcup_\alpha X_{\mu_\alpha}=\varnothing$. Since $\mu_\alpha(X_{\mu_\alpha})=1$ for all α, we have $\mu(\bigcup_\alpha X_{\mu_\alpha})=1$ and hence $\mu(X_\mu)=0$.

sufficiently low temperature there is an infinite family of (translation invariant) Gibbs measures; for two of them $\mu(X_\mu)=1$, for all the others $\mu(X_\mu)=0$, but the dimension $D_\lambda(X_\mu^{(T)})$ is the same for all of them. (We are indebted to the referee for calling our attention to this corollary).

The paper is organized as follows. In Section 2 we provide the information used in the subsequent proofs. We, in particular, state, for $d \geqslant 1$, a refined version of Breiman's theorem for generic points of Gibbs measures; this result, apparently being of some interest by itself, will be one of our main tools in the sequel. We also give some important examples of "stable scale metrics" with respect to which the dimension $D_\lambda(X_\mu^{(T)})$ is evaluated. Our main results are stated and partially proved in Section 3. If μ is not ergodic the evaluation of $D_\lambda(X_\mu^{(T)})$ is based on employing an auxiliary measure $\tilde{\mu}$ constructed in Section 4.

The following notation is used throughout the paper: $\mathscr{F}^{(T)}$ denotes the set of all finite subsets of T; $\mathscr{P}^{(T)}$ is the set of Borel probability measures on $X^{(T)}$; $\mathscr{I}^{(T)}$ the set of τ-invariant measures in $\mathscr{P}^{(T)}$; $\mathscr{E}^{(T)}$ the set of ergodic measures in $\mathscr{I}^{(T)}$. For any $A \subset T$ and $x \in X^{(T)}$ we denote by x_A the restriction of x to A. The set $C_V^{(T)}(x)=\{y\colon y \in X^{(T)},\, y_V=x_V\}$ where $V \subset T$ is called the cylinder with support V; we denote by $\mathscr{C}_V^{(T)}$ the set of all such cylinders. If $\{A_n\}$ is a specified sequence, $C_n(x)$ denotes the cylinder $C_V(x)$ when $V=A_n$. We write $\hat{X}$, $\hat{\mathscr{F}}$, $\hat{\mathscr{P}}$, etc., when $T=\mathbb{Z}^d$. We denote by $\hat{\mathscr{G}}$ the set of all τ-invariant Gibbs measures on $\hat{X}=X^{(\mathbb{Z}^d)}$ and by $\hat{\mathscr{G}}(U)$ the set of measures in $\hat{\mathscr{G}}$ corresponding to a specific potential U. The projections of these sets to $X^{(T)}$ will be denoted by $\mathscr{G}^{(T)}$ and $\mathscr{G}^{(T)}(U)$, respectively. We shall often drop the index T as well as the hat if there is no danger of confusion.

If $A, B \subset \mathbb{Z}^d$ and $\mathbf{t} \in \mathbb{Z}^d$ we set $\bar{A}=\mathbb{Z}^d \setminus A$, $A-B=\{\mathbf{s}:\mathbf{s}=\mathbf{a}-\mathbf{b},\, \mathbf{a} \in A,\, \mathbf{b} \in B\}$, $\mathbf{t}+A=\{\mathbf{s}:\mathbf{s}=\mathbf{t}+\mathbf{a},\, \mathbf{a} \in A\}$.

Some results of this paper were announced in ref. 7.

2. AUXILIARY RESULTS

2.1. Refined Breiman Type Theorems

Unless the contrary is stated, in this subsection we put $T=\mathbb{Z}^d$ and drop the hat in the notation. Let $\mathscr{A}=\{A_n\}$ be a sequence of finite subsets of $\mathbb{Z}^d$. $\mathscr{A}$ is called a *van Hove sequence* if $\lim_{n\to\infty} |A_n|^{-1}\, |A_n \,\Delta(\mathbf{t}+A_n)|=0$ for each $\mathbf{t} \in \mathbb{Z}^d$. Denote: $A_n'=\bigcup_{k\leqslant n} A_n$. $\mathscr{A}$ is called a *regular sequence* if there is a constant M such that $|A_n'-A_n| \leqslant M|A_n|$, $n=1, 2,\ldots$.

We fix a τ-invariant potential U and consider the pressure $P(U)$ and the set $\mathscr{G}(U)$ of Gibbs measures corresponding to U (see, e.g., refs. 4 and 11). In refs. 3, 12, and 13 (see also ref. 14, Theorems 8.3.1 and 8.7.3) pointwise convergence theorems related to the specific entropy have been proved (see Theorem 2.2 below). Using the same approach as in ref. 13, one can obtain the following refined version of these theorems (Theorem 2.1 below). Denote

$$\varphi(x)=\varphi_U(x)=\sum_{B\in\mathscr{F}:\mathbf{0}\in B}|B|^{-1}\,U(B,x),\quad x\in X,$$

and for any $\mu\in\mathscr{I}$ consider the mean energy $e_\mu(U):=\mathbf{E}_\mu\varphi$. The function φ is clearly continuous on X. Let

$$X_{\varphi,\mu}=\left\{x\in X:\lim_{n\to\infty}|A_n|^{-1}\sum_{\mathbf{t}\in A_n}\varphi(\tau_{\mathbf{t}}x)=\mathbf{E}_\mu\varphi\right\}.$$

Theorem 2.1. Let $\mathscr{A}$ be a van Hove sequence. If $\mu\in\mathscr{I}$, $\nu\in\mathscr{G}(U)$ then

(a) for any $x\in X_{\varphi,\mu}$ there is the limit

$$\lim_{n\to\infty}\left[-|A_n|^{-1}\ln\nu(C_n(x))\right]=:h(\nu,\mu);$$

(b) $h(\nu,\mu)=e_\mu(U)+P(U)$.

The proof of this theorem will be published elsewhere.

Remark 2.1. By the variational principle (see refs. 4 and 11) $h(\nu,\mu)=h(\nu)$ if $\mu=\nu$.

Theorem 2.1 implies the following statement (see refs. 3, 12–14).

Theorem 2.2. Let $\{A_n\}$ be a regular van Hove sequence. If $\nu\in\mathscr{G}$, $\mu\in\mathscr{E}$ then

$$\lim_{n\to\infty}\left[-|A_n|^{-1}\ln\nu(C_n(x))\right]=h(\nu,\mu),\quad \mu-\text{a.e.}$$

Let T be a semigroup in $\mathbb{Z}^d$. Now we are interested in van Hove sequences $\{A_n\}$ with $A_n\subset T$, $n=1,2,\ldots$; we will say that such sequences are *van Hove in T*. We say that T is a *massive subsemigroup* of $\mathbb{Z}^d$ if for any $F\in\hat{\mathscr{F}}$ there is a vector $\mathbf{t}\in\mathbb{Z}^d$ such that $F+\mathbf{t}\subset T$. It is easy to verify that a subsemigroup is massive if and only if it contains at least one van Hove sequence.

Let T be a massive semigroup in $\mathbb{Z}^d$ and $\mathscr{A} = \{A_n\}$ a van Hove sequence in T. If $\mu \in \mathscr{I}^{(T)}$, we denote its (unique) τ-invariant extension to $\hat{X}$ by $\hat{\mu}$. It is easy to check that if $x \in X_\mu^{(T)}(\mathscr{A})$ then each extension $\hat{x}$ of x to $\mathbb{Z}^d$ belongs to $\hat{X}_{\hat{\mu}}(\mathscr{A})$. Besides, $\hat{\mu}(\hat{C}_n(\hat{x})) = \mu(C_n^{(T)}(x))$. Recall that $\mu \in \mathscr{G}^{(T)}(U)$ means, by definition, $\hat{\mu} \in \hat{\mathscr{G}}(U)$. These remarks bring us to the following corollary of Theorems 2.1 and 2.2.

Corollary 2.1. (a) If $\mathscr{A}$ is a van Hove sequence and $\mu \in \mathscr{I}^{(T)}$, $\nu \in \mathscr{G}^{(T)}(U)$, then for every $x \in X_\mu^{(T)}(\mathscr{A})$

$$\lim_{n \to \infty} [-|A_n|^{-1} \ln \nu(C_n^{(T)}(x))] = e_{\hat{\mu}}(U) + P(U) = h(\hat{\nu}, \hat{\mu}). \tag{2.1}$$

(b) If $\mathscr{A}$ is a regular van Hove sequence and $\mu \in \mathscr{E}^{(T)}$, then (2.1) holds μ-a.e.

2.2. Scale Metrics and Evaluation of the Hausdorff Dimension

Let $X = S^T$, where S is finite and T is infinite countable. The notion of a scale metric on X was introduced in ref. 15 (see also refs. 5–7). By definition, every scale metric is in some sense compatible with a scale sequence $\mathbf{a}$ defined as follows. Let $\mathscr{A} = \{A_n\}$ be a sequence of finite subsets of T such that $A_n \uparrow T$ and $\lim_{n \to \infty} |A_{n+1}|/|A_n| = 1$. A sequence $\mathbf{a} = \{a_n, n \geqslant 1\}$ of functions $a_n \colon X \to \mathbb{R}$ is called a *scale sequence* if

(A) $a_n(x) > 0$ and $a_n(x) \downarrow 0$ for every $x \in X$;

(B) $a_n(x)$ is constant on every cylinder $C_n(x) = \{y \colon y \in X, y_{A_n} = x_{A_n}\}$, $x \in X$.

Let $W \subseteq X$. We say that a scale sequence $\mathbf{a}$ is *stable on W* (or W-stable)[4] if for each $x \in W$ the following limit exists and is a positive constant on W:

$$\lim_{n \to \infty} [-|A_n|^{-1} \ln a_n(x)] =: \kappa(W), \qquad x \in W. \tag{2.2}$$

Example 2.1. If $0 < \theta < 1$ and $a_n(x) \equiv \theta^{|A_n|}$ then the functions a_n form an X-stable scale sequence with $\kappa(X) = -\ln \theta$.

Instead of repeating here the definition of a scale metric compatible with a scale sequence $\mathbf{a}$ (every such a metric is called an $\mathbf{a}$-metric, and it is called W-stable if $\mathbf{a}$ is W-stable) we give some important examples of scale metrics.

[4] In ref. 15 another term was used instead of "stability."

Example 2.2 (Standard a-Metrics). With each scale sequence $\mathbf{a} = \{a_n\}$ a *standard* **a**-*metric* $\lambda_{\mathbf{a}}$ can be associated in the following way: for any $x, y \in X, x \neq y$, we put $n(x, y) = \min\{n : x_{A_n} \neq y_{A_n}\}$; then $\lambda_{\mathbf{a}}(x, y) = 1$ if $n(x, y) = 1$, and $\lambda_{\mathbf{a}}(x, y) = a_{n(x, y)-1}(x)$ otherwise. It is evident that $\operatorname{diam}(C_n(x)) = a_n(x)$ and $\lambda(x, y) > a_n(x)$ if $y \notin C_n(x)$.

Example 2.3 (Billingsley Metrics). Let U be a potential and $\nu \in \mathscr{G}(U)$; then $\mathbf{a} := \{\nu(C_n(x))\}$ is a scale sequence. The standard **a**-metric (we denote it by λ_ν) is called the *Billingsley metric* associated with ν; in other words, if $x \neq y$ then $\lambda_\nu(x, y) = \nu(C_{n(x, y)-1}(x))$. If $\mathscr{A}$ is a van Hove sequence then, for any measure $\mu \in \mathscr{I}$, λ_ν is $X_{\varphi, \mu}$-stable with $\kappa(X_{\varphi, \mu}) = h(\nu, \mu)$ where $\varphi(x) = \varphi_U(x)$ is defined in Subsection 2.1. The Hausdorff dimension with respect to λ_ν is often called the *Billingsley dimension* with respect to ν.

Example 2.4 (Potential Metrics). These are **a**-metrics with $a_n(x) = \exp[-E_n(x)]$, where $E_n(x)$ is the energy (corresponding to a potential U) of the configuration x in the "vessel" A_n (see ref. 5 for details). A potential metric λ is $X_{\varphi, \mu}$-stable and $\kappa(X_{\varphi, \mu}) = e_\mu(U)$ for any $\mu \in \mathscr{I}$ (here $\varphi = \varphi_U$, see before). A particular case of such a metric (for $T = \mathbb{Z}^+$, $A_n = [0, n-1] \cap T$) is well known: it corresponds to a one-particle potential $U(s) \equiv -\ln \theta > 0$, $s \in S$ (the zero interaction), and is the standard **a**-metric (see Example 2.2) with $a_n(x) \equiv \theta^n$.

Let **a** be a scale sequence and λ an **a**-metric. With any measure $\mu \in \mathscr{P}$ one can associate the *upper* and *lower local dimensions* as follows:

$$\bar{d}_{\lambda, \mu}(x) := \limsup_{n \to \infty} \frac{\ln \mu(C_n(x))}{\ln a_n(x)}; \quad \underline{d}_{\lambda, \mu}(x) := \liminf_{n \to \infty} \frac{\ln \mu(C_n(x))}{\ln a_n(x)}, \quad x \in X.$$

The following Billingsley type theorem presents a convenient tool for evaluation of the Hausdorff dimension with respect to a scale metric; it is a generalization of Theorem 1.6 in ref. 15 and can be proved similarly. Recall that $D_\lambda(F)$ is the Hausdorff dimension of a set $F \subseteq X$ with respect to λ.

Theorem 2.3. Let $F \subseteq X$, $\bar{d}$, $\underline{d}$ be constants, $\mu \in \mathscr{P}$ and let μ^* be the corresponding outer measure.

(1) If $\bar{d}_{\lambda, \mu}(x) \leqslant \bar{d}$ for all $x \in F$, then $D_\lambda(F) \leqslant \bar{d}$.

(2) If λ is F-stable, $\mu^*(F) > 0$ and $\underline{d}_{\lambda, \mu}(x) \geqslant \underline{d}$ for all $x \in F$, then $D_\lambda(F) \geqslant \underline{d}$.

2.3. Two Properties of Gibbs Measures

It is known (see (16.38) in ref. 4 and references therein) that for any potential U and any $\nu \in \hat{\mathscr{G}}(U)$ there exists a sequence of ergodic measures $\nu_i \in \hat{\mathscr{G}}(U_i)$ such that ν_i weakly converges to ν and $\|U_i - U\| \to 0$. This implies that $P(U_i) \to P(U)$, $e_{\nu_i}(U_i) \to e_\nu(U)$ and hence, due to the variational principle, $h(\nu_i) \to h(\nu)$. Let now T be a massive semigroup and $\mu \in \mathscr{G}^{(T)}(U)$. Then $\hat{\mu}$, the invariant extension of μ to $\hat{X}$, belongs to $\hat{\mathscr{G}}(U)$ and $h(\mu) = h(\hat{\mu})$. The following simple corollary of these facts plays a crucial role in Section 4.

Theorem 2.4. For each $\mu \in \mathscr{G}^{(T)}$ there exists a sequence of measures $\mu_i \in \mathscr{G}^{(T)} \cap \mathscr{E}^{(T)}$ such that μ_i weakly converges to μ and $h(\mu_i) \to h(\mu)$.

The next assertion is also used in Section 4. It can be easily deduced from the definition of a Gibbs measure (cf. the proof of Theorem 8.7.3 in ref. 14).

Theorem 2.5. Let $\mu \in G^{(T)}$ and $\{A_n\}$ a van Hove sequence in T. There is a sequence of non-negative numbers $u_n \to 0$ such that if $C \in \mathscr{C}^{(T)}_{A_n}$, $V \in \mathscr{F}^{(T)}$, $V \subset T \setminus A_n$, $C' \in \mathscr{C}^{(T)}_V$, then

$$\mu(C)\,\mu(C')\,e^{-u_n |A_n|} \leqslant \mu(C \cap C') \leqslant \mu(C)\,\mu(C')\,e^{u_n |A_n|}, \qquad n \in \mathbb{N}.$$

3. MAIN RESULTS: HAUSDORFF DIMENSION OF X_μ

Let T be a semigroup in $\mathbb{Z}^d$ and $X = X^{(T)}$; in this section we drop the index (T) unless the contrary is stated. We assume now that T is a massive semigroup. Let $\mathscr{T} = \{T_n\}$ be a regular van Hove sequence in T, $\bigcup_n T_n = T$, and let $X_\mu = X_\mu(\mathscr{T})$ denote the set of μ-generic points with respect to this sequence. In what follows $\mathscr{A} = \{A_n\}$ is a subsequence of $\mathscr{T}$, $A_n \uparrow T$, $\lim_{n\to\infty} |A_{n+1}|/|A_n| = 1$, and λ is an X_μ-stable scale metric (with respect to $\mathscr{A}$). Recall that if $\mu \in \mathscr{I}$ we denote by $\hat{\mu}$ the τ-invariant extension of μ to $\hat{X}$ and that $\hat{\mu} \in \hat{\mathscr{G}}(U)$ if $\mu \in \mathscr{G}(U)$.

Theorem 3.1. If U is a potential and $\mu \in \mathscr{E} \cap \mathscr{G}(U)$ then

$$D_\lambda(X_\mu) = \frac{h(\mu)}{\kappa(X_\mu)} = \frac{e_{\hat{\mu}}(U) + P(U)}{\kappa(X_\mu)}. \tag{3.1}$$

This is a simple corollary of Theorems 2.1 and 2.3 and the obvious inclusion $X_\mu \subset X_{\varphi,\mu}$ (see Subsection 2.1)

When μ is not ergodic (and hence $\mu(X_\mu) = 0$) the evaluation of $D_\lambda(X_\mu)$ is an essentially more challenging problem. Nevertheless an upper bound can be readily deduced from Theorem 2.3 just in the same way as the previous theorem.

Theorem 3.2. If U is a potential and $\mu \in \mathcal{G}(U)$ then

$$D_\lambda(X_\mu) \leqslant \frac{h(\mu)}{\kappa(X_\mu)} = \frac{e_{\hat{\mu}}(U) + P(U)}{\kappa(X_\mu)}. \tag{3.2}$$

But the lower estimate cannot be obtained from Theorem 2.3 in such a straightforward way as the upper one. Because of this we shall construct another probability measure $\hat{\mu}$ concentrated on X_μ (but, in general, not invariant anymore) to which this theorem can be applied. However, unlike in the case $d = 1$, we have to restrict our study to Gibbs measures because just for such measures the desired construction is available.

For the sake of simplicity we shall consider only one massive semigroup $T \subset \mathbb{Z}^d$, namely $T = \mathbb{Z}_+^d$ (see also Remark 3.1).

With any $\mathbf{s} = (s^1, ..., s^d) \in \mathbb{Z}_+^d$ we associate the parallelepiped $\Pi = \Pi(\mathbf{s}) = \prod_{i=1}^d [0, s^i - 1] \cap \mathbb{Z}^d$ and denote: $\underline{l}(\Pi) = \min_i s^i$, $\bar{l}(\Pi) = \max_i s^i$. Let Par be the set all such parallelepipeds. A sequence $\{T_n\}$ of parallelepipeds in Par is van Hove if and only if $\underline{l}(T_n) \to \infty$.

Let $1 \leqslant c < \infty$. We denote by Par(c) the family of all parallelepipeds $\Pi \in$ Par such that $\bar{l}(\Pi)/\underline{l}(\Pi) \leqslant c$. A sequence of parallelepipeds $\{T_n\}$ is regular if and only if $\{T_n\} \subset$ Par(c) for some c.

In the next section we prove the following lemma that plays a crucial role in the evaluation of the desired lower bound for the Hausdorff dimension $D_\lambda(X_\mu)$.

Lemma 3.1. Let $\mathcal{T} = \{T_n\}$ be a regular van Hove sequence of parallelepipeds in Par. Then for any measure $\mu \in \mathcal{G}$ there exists a measure $\tilde{\mu} \in \mathcal{P}$ concentrated on X_μ and such that for all $x \in X_\mu$

$$\liminf_{n \to \infty} [-|T_n|^{-1} \ln \tilde{\mu}(C_n(x))] \geqslant h(\mu). \tag{3.3}$$

This lemma and Theorem 2.3 bring us immediately to the following statement.

Theorem 3.3. Let U be a potential, $\mu \in \mathcal{G}(U)$; let $\mathcal{T} = \{T_n\}$ be as in Lemma 3.1, $\mathcal{A} = \{A_n\}$ a subsequence of $\mathcal{T}$, $A_n \uparrow T$, and λ a X_μ-stable scale metric (with respect to $\mathcal{A}$). Then

$$D_\lambda(X_\mu) \geqslant \frac{h(\mu)}{\kappa(X_\mu)} = \frac{e_{\hat{\mu}}(U) + P(U)}{\kappa(X_\mu)}.$$

Theorems 3.2 and 3.3 imply our main result.

Theorem 3.4. Formula (3.1) holds under the conditions of Theorem 3.3.

Remark 3.1. Let us define a class of sub-semigroups of $\mathbb{Z}^d$ including $\mathbb{N}^d$, $\mathbb{Z}_+^d$, and $\mathbb{Z}^d$. For any $i = 1,..., d$ we put $\mathbf{e}_i = (e_i^1,..., e_i^d)$ where $e_i^i = 1$ and $e_i^j = 0$ if $j \neq i$ $(i, j = 1,..., d)$. We fix a vector $\mathbf{c} = (c^1,..., c^d) \in \mathbb{Z}_+^d$ and two sets, $I, J \subset \{1,..., d\}$, such that $I \cup J = \{1,..., d\}$, and say that T is an *axial semigroup* in $\mathbb{Z}^d$ specified by $\mathbf{c}$ and I, J if it consists of all vectors of the form $\sum_{i \in I} (a^i + c^i)\, \mathbf{e}_i - \sum_{j \in J} (b^j + c^j)\, \mathbf{e}_j$ where $a^i, b^j \in \mathbb{Z}_+$. It is easy to check that any axial semigroup is massive. Theorem 3.4 can be easily extended to such a semigroup (it is quite possible that it can be extended also to all massive semigroups).

4. PROOF OF LEMMA 3.1

We use the notation introduced in Sections 1–3; in particular, $\{T_n\}$ is a fixed sequence in $T = \mathbb{Z}_+^d$ consisting of parallelepipeds $T_n \in \operatorname{Par}(c)$ such that $\lim_{n \to \infty} \underline{l}(T_n) = \infty$ (since no subsemigroups of $\mathbb{Z}^d$ different from $T = \mathbb{Z}_+^d$ will appear, we drop the upper index (T) in the notation).

For any vector $\mathbf{t} = (t^1,..., t^d) \in \mathbb{Z}^d$ with $t^k \geqslant 1$, $k = 1,..., d$, we set

$$\Pi(\mathbf{t}) = \Pi(t^1,..., t^d) = \mathbb{Z}^d \cap \prod_{k=1}^{d} [0, t^k - 1].$$

In particular, $T_n = \Pi(\mathbf{t}_n)$ for some $\mathbf{t} \in T$, $n = 1, 2,...$.

We introduce a metric ρ on $\mathscr{P}$ by the formula

$$\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{C \in \mathscr{C}_{K(n)}} |\nu(C) - \nu'(C)|, \qquad \nu, \nu' \in \mathscr{P}, \tag{4.1}$$

where $\mathscr{C}_{K(n)}$ is the set of cylinders with support $K(n) = [0, n-1]^d \cap \mathbb{Z}^d$ (this metric for $d = 1$ was used in ref. 1). It is easy to see that the convergence in the metric space $(\mathscr{P}, \rho)$ is equivalent to the weak convergence of measures. One can also check that for any probability vector $p = (p_1,..., p_k)$ and any measures $\nu_j, \nu_j' \in \mathscr{P}$, $j = 1,..., k$, we have

$$\rho\left(\sum_{j=1}^{k} p_j \nu_j, \sum_{j=1}^{k} p_j \nu_j'\right) \leqslant \sum_{j=1}^{k} p_j \rho(\nu_j, \nu_j'). \tag{4.2}$$

The subsequent reasoning is divided into several steps.

1. Construction of the Measure $\tilde{\mu}$. We choose four sequences of positive numbers, γ_i, ε_i, η_i, l_i ($i = 1, 2,...$), such that l_i, $1/\eta_i \in \mathbb{Z}$ and

$$\text{(a)}\quad \lim_{i\to\infty} \gamma_i = \lim_{i\to\infty} \eta_i = 0, \qquad \text{(b)}\quad \sum_{i=1}^{\infty} \varepsilon_i < \infty, \qquad \text{(c)}\quad \lim_{i\to\infty} l_i = \infty, \tag{4.3}$$

$$\frac{2}{\eta_i} \leqslant \frac{1}{l_i+1} \left(\frac{\varepsilon_i}{\gamma_i}\right)^{1/d}, \qquad i = 1, 2,\dots . \tag{4.4}$$

To make sure that such sequences do exist it suffices to set

$$\gamma_i = 1/2^d i^{d+3}, \quad \varepsilon_i = 1/i^2, \quad \eta_i = 1/i, \quad l_i = [i^{1/2d}] - 1.$$

For each $x \in X$ we denote by δ^x the probability measure on X concentrated at the point x and let

$$\delta_V^x = \frac{1}{|V|} \sum_{t\in V} \delta^{\tau_t x}, \qquad x \in X, \quad V \in \mathscr{F}. \tag{4.5}$$

Since $\mu \in \mathscr{G}$, there is a sequence of ergodic measures $\mu_i \in \mathscr{G}$ ($i \geqslant 0$) such that

$$\lim_{i\to\infty} \rho(\mu_i, \mu) = 0, \qquad \lim_{i\to\infty} h(\mu_i) = h(\mu) \tag{4.6}$$

(see Theorem 2.4).

From the pointwise ergodic theorem (see Theorem 6.3.1 in ref. 14) related to averaging over parallelepipeds and the Egorov theorem one can easily deduce the following: for every $i \geqslant 1$ there exist a Borel set $M_i \subset X$ and a positive integer m_i such that

$$\mu_i(M_i) \geqslant 1 - \gamma_i \tag{4.7}$$

and for all $x \in M_i$ and all $\Pi \in \mathrm{Par}(c)$ with $\underline{l}(\Pi) \geqslant m_i$ (recall that $\underline{l}(\Pi)$ is the minimal edge of Π) we have

$$\rho(\delta_\Pi^x, \mu_i) \leqslant \eta_i, \tag{4.8}$$

$$\left| \frac{1}{|\Pi|} \ln \mu_i(C_\Pi(x)) + h(\mu_i) \right| \leqslant \eta_i. \tag{4.9}$$

According to Theorem 2.5 for every i there is a sequence $u_n^{(i)} \to 0$ ($n \to \infty$) such that

$$\mu_i(C_{K(n)}(x) \cap C_V(x)) \leqslant \mu_i(C_{K(n)}(x))\, \mu_i(C_V(x)) \exp(u_n^{(i)} \cdot |K(n)|), \tag{4.10}$$

whenever $V \in \mathscr{F}, V \subset T \setminus K(n)$. Therefore, we can find a number b_i such that $n \geqslant b_i$ implies $u_n^{(i)} \leqslant \eta_i$ $(i = 1, 2,...)$. We set $k_1 = \max\{m_1, b_1\}$ and choose a sequences of positive integers k_i such that for all $i \geqslant 1$

$$k_{i+1} > \max\left\{m_{i+1}, b_{i+1}, k_i + 1, \frac{2k_i \eta_{i+1}}{\eta_i^2}, \frac{2k_i (l_i + 1)\, \eta_{i+1}}{\eta_i}\right\}. \tag{4.11}$$

We also set

$$r_i = 2k_i / \eta_i. \tag{4.12}$$

From (4.3), (4.11), and (4.12) it follows that

$$r_i / r_{i+1} \leqslant \eta_i, \qquad \lim_{i \to \infty} k_i = \lim_{i \to \infty} r_{i+1} / r_i = \infty. \tag{4.13}$$

Denote by $\mathbf{r}_i$ the vector in $\mathbb{Z}^d$ with all coordinates equal to r_i. For every vector $\mathbf{u} = (u^1,..., u^d)$ we set $\bar{u} = \max_k u^k$, $\underline{u} = \min_k u^k$.

Let

$$D_i = K((l_i + 1)\, r_i) \setminus K(r_i), \qquad L_i = D_i \cap k_i \mathbb{Z}^d, \quad i \geqslant 1. \tag{4.14}$$

Clearly,

$$|L_i| \leqslant \left(\frac{(l_i + 1)\, r_i}{k_i}\right)^d = \left(\frac{2(l_i + 1)}{\eta_i}\right)^d. \tag{4.15}$$

For each $M \subset X$ and $V \in \mathscr{F}$ we denote

$$M_V = \bigcup_{x \in M} C_V(x).$$

Let now

$$\tilde{M}_i = \bigcap_{\mathbf{t} \in L_i} (\tau_{-\mathbf{t}} M_i)_{\mathbf{t} + K(r_{i+1} - \bar{t})}, \tag{4.16}$$

$$\tilde{M} = \liminf_{i \to \infty} \tilde{M}_i = \bigcup_j \bigcap_{i \geqslant j} \tilde{M}_i. \tag{4.17}$$

We determine the measure $\tilde{\mu}$ (see the statement of Lemma 3.1) by the following requirements:

(a) at every cylinder C with support $K(r_{i+1}) \setminus K(r_i)$ $(i = 0, 1,...,$ where we let $K(r_0) = \varnothing)$ the measures $\tilde{\mu}$ and μ_i coincide;

(b) the cylinders mentioned in (a) and corresponding to different i's are $\tilde{\mu}$-independent.

2. Estimation of $\tilde{\mu}(\tilde{M})$. We shall prove that $\tilde{\mu}(\tilde{M})=1$. By the Borel–Cantelli Lemma it suffices to check that

$$\sum_i (1-\tilde{\mu}(\tilde{M}_i))<\infty \tag{4.18}$$

(see (4.17)). By (4.16) for all $i\geqslant 1$ the set $\tilde{M}_i$ is a union of cylinders with support $K(r_{i+1})\setminus K(r_i)$, so the definition of $\tilde{\mu}$ implies that $\tilde{\mu}(\tilde{M}_i)=\mu_i(\tilde{M}_i)$. Due to (4.7) and the τ-invariance of μ_i we have

$$\begin{aligned}1-\mu_i(\tilde{M}_i)&\leqslant 1-\mu_i\left(\bigcap_{t\in L_i}\tau_{-\mathbf{t}}M_i\right)\\&\leqslant\sum_{\mathbf{t}\in L_i}(1-\mu_i(\tau_{-\mathbf{t}}M_i))=\sum_{\mathbf{t}\in L_i}(1-\mu_i(M_i))\leqslant\gamma_i\,|L_i|.\end{aligned}$$

Therefore (see (4.15))

$$1-\tilde{\mu}(\tilde{M}_i)=1-\mu_i(\tilde{M}_i)\leqslant\gamma_i\left(\frac{2(l_i+1)}{\eta_i}\right)^d,$$

and from (4.4) we obtain $1-\mu_i(\tilde{M}_i)\leqslant\varepsilon_i$, which together with (4.3) leads to (4.18).

3. Auxiliary Statements. We are going to prove that $\tilde{M}\subset X_\mu$ and hence $\tilde{\mu}$ is concentrated on X_μ. Since the convergence in the metric space $(\mathscr{P},\rho)$ is the weak convergence of measures, it suffices to show that for every $x\in\tilde{M}$

$$\lim_{n\to\infty}\rho(\delta^x_{T_n},\mu)=0,$$

where $\delta^x_{T_n}$ is the "empirical measure" defined according to (4.5). We start with a general statement.

Lemma 4.1. Let $x,y\in X$ and $\Pi=\Pi(\mathbf{s})$ be a parallelepiped with $\mathbf{s}=(s^1,\ldots,s^d)\in\mathbb{Z}^d_+$, $\underline{s}>2$. Assume that $x(\mathbf{t})=y(\mathbf{t})$ for any $\mathbf{t}\in K(\bar{s})$. Then $\rho(\delta^x_\Pi,\delta^y_\Pi)\leqslant 2d/\underline{s}$.

Proof. If $j\leqslant\bar{s}$ and $C\in\mathscr{C}_{K(j)}$ then $|\delta^x_\Pi(C)-\delta^y_\Pi(C)|=0$ for $j=1$ and

$$|\delta^x_\Pi(C)-\delta^y_\Pi(C)|\leqslant|\Pi|^{-1}\,(|\Pi|-|\Pi(s^1-j+1,\ldots,s^d-j+1)|)\qquad\text{for}\quad j>1. \tag{4.19}$$

Let $a_k > 0$, $0 \leqslant c_k \leqslant 1$ $(k = 1,..., d)$, and $0 \leqslant b \leqslant \min_k a_k$. The simple-to-prove inequalities

$$1 - \prod_{k=1}^{d} (1 - c_k) \leqslant 1 - (1 - \bar{c})^d \leqslant d \max_k c_k$$

with $c_k = b/a_k$ imply

$$\left(\prod_{k=1}^{d} a_k\right)^{-1} \left(\prod_{k=1}^{d} a_k - \prod_{k=1}^{d} (a_k - b)\right) \leqslant \frac{db}{\min_k a_k}. \tag{4.20}$$

By (4.20) we can change the right-hand side of (4.19) for $dj/\underline{s}$ and obtain (see (4.1))

$$\rho(\delta_{\Pi}^{x}, \delta_{\Pi}^{y}) \leqslant \sum_{j=2}^{\bar{s}} \frac{dj}{2^j \underline{s}} + \frac{1}{2^{\underline{s}}} \leqslant \frac{d}{\underline{s}} \sum_{j=2}^{\infty} \frac{j}{2^j} + \frac{1}{2^{\underline{s}}} = \frac{2d}{\underline{s}}. \quad \blacksquare$$

In the sequel we shall repeatedly use the following simple fact: if $V = \bigsqcup_{j=1}^{k} V_j$, $V_j \in \mathscr{F}$, then

$$\delta_V^x = \sum_{j=1}^{k} \frac{|V_j|}{|V|} \delta_{V_j}^x, \qquad x \in X. \tag{4.21}$$

Lemma 4.2. Let $\Pi_i \in \mathrm{Par}(c)$, $\mathbf{t}_i \in L_i$ (see (4.14)), $\mathbf{t}_i + \Pi_i \subset K(r_{i+1})$, and $\underline{l}(\Pi_i) \to \infty$ as $i \to \infty$. Then for every $x \in \tilde{M}$

$$\lim_{i \to \infty} \rho(\delta_{\mathbf{t}_i + \Pi_i}^x, \mu) = 0, \tag{4.22}$$

$$\liminf_{i \to \infty} \left[-\frac{\ln \mu_i(C_{\mathbf{t}_i + \Pi_i}(x))}{|\Pi_i|} \right] \geqslant h(\mu). \tag{4.23}$$

Proof. By (4.16), (4.17) $x \in \tilde{M}_i$ if i is sufficiently large. For such an i and any $\mathbf{t} \in L_i$ there is a point $y = y_{x,i,\mathbf{t}} \in \tau_{-\mathbf{t}} M_i$ such that $x(\mathbf{s}) = y(\mathbf{s})$ for all $\mathbf{s} \in \mathbf{t} + K(r_{i+1} - \bar{t})$. It follows that $\tau_{\mathbf{t}}\, y \in M_i$ and $(\tau_{\mathbf{t}} x)(\mathbf{s}) = (\tau_{\mathbf{t}}\, y)(\mathbf{s})$ for all $\mathbf{s} \in K(r_{i+1} - r_i)$ (see (4.16), (4.17)). In our case we take $y = y_{x,i,\mathbf{t}_i}$ and note that $x(\mathbf{s}) = y(\mathbf{s})$ for all $\mathbf{s} \in \mathbf{t}_i + K(\bar{l}(\Pi_i))$. Besides, $\delta_{\mathbf{t}_i + \Pi_i}^x = \delta_{\Pi_i}^{\tau_{\mathbf{t}_i} x}$, $\delta_{\mathbf{t}_i + \Pi_i}^y = \delta_{\Pi_i}^{\tau_{\mathbf{t}_i} y}$. Using this fact together with the triangle inequality, we obtain

$$\begin{aligned}\rho(\delta_{\mathbf{t}_i + \Pi_i}^x, \mu) &\leqslant \rho(\delta_{\mathbf{t}_i + \Pi_i}^x, \delta_{\mathbf{t}_i + \Pi_i}^y) + \rho(\delta_{\mathbf{t}_i + \Pi_i}^y, \mu_i) + \rho(\mu_i, \mu) \\ &= \rho(\delta_{\Pi_i}^{\tau_{\mathbf{t}_i} x}, \delta_{\Pi_i}^{\tau_{\mathbf{t}_i} y}) + \rho(\delta_{\Pi_i}^{\tau_{\mathbf{t}_i} y}, \mu_i) + \rho(\mu_i, \mu). \end{aligned} \tag{4.24}$$

To estimate the first and the second terms on the right-hand side of (4.24) we can apply Lemma 4.1 with $\Pi = \Pi_i$, $x = \tau_{\mathbf{t}_i} y$ and inequality (4.8) with $\Pi = \Pi_i$, respectively. This gives

$$\rho(\delta^x_{\mathbf{t}_i+\Pi_i}, \mu) \leqslant 2d/\underline{l}(\Pi_i) + \eta_i + \rho(\mu_i, \mu), \tag{4.25}$$

and we come to (4.22) (see (4.3), (4.6)).

In order to prove (4.23) let us note that $C_{\mathbf{t}_i+\Pi_i}(x) = C_{\mathbf{t}_i+\Pi_i}(y)$ where $y = y_{x,i,\mathbf{t}_i}$. Therefore, in view of the τ-invariance of μ_i and (4.9),

$$\frac{\ln \mu_i(C_{\mathbf{t}_i+\Pi_i}(x))}{|\Pi_i|} = \frac{\ln \mu_i(C_{\mathbf{t}_i+\Pi_i}(y))}{|\Pi_i|} = \frac{\ln \mu_i(C_{\Pi_i}(\tau_{\mathbf{t}_i} y))}{|\Pi_i|} \leqslant -h(\mu_i) + \eta_i.$$

Since $\eta_i \to 0$ we obtain (4.23). ∎

The following obvious corollary will be more convenient for references.

Corollary 4.1. If $\{i(n)\}$ is an $\mathbb{N}$-valued sequence such that $i(n) \to \infty$ as $n \to \infty$, and if $\mathbf{t}_{i(n)}$ and $\Pi_{i(n)}$ are defined as in Lemma 4.2, then for every $x \in \tilde{M}$

$$\lim_{n\to\infty} \rho(\delta^x_{\mathbf{t}_{i(n)}+\Pi_{i(n)}}, \mu) = 0, \qquad \liminf_{n\to\infty} \left[-\frac{\ln \mu_{i(n)}(C_{\mathbf{t}_{i(n)}+\Pi_{i(n)}}(x))}{|\Pi_{i(n)}|} \right] \geqslant h(\mu).$$

Lemma 4.3. Let

$$\Lambda_i \subset L_i, \quad \Lambda'_i = \bigcup_{\mathbf{t}\in\Lambda_i} (\mathbf{t} + K(k_i)), \qquad i = 1, 2,\ldots \tag{4.26}$$

Then for every $x \in \tilde{M}$

$$\lim_{i\to\infty} \rho(\delta^x_{\Lambda'_i}, \mu) = 0, \tag{4.27}$$

$$\liminf_{i\to\infty} \left[-\frac{\ln \mu_i(C_{\Lambda'_i}(x))}{|\Lambda'_i|} \right] \geqslant h(\mu). \tag{4.28}$$

Proof. By (4.21) and (4.2)

$$\rho(\delta^x_{\Lambda'_i}, \mu) \leqslant \sum_{\mathbf{t}\in\Lambda_i} \frac{|\mathbf{t}+K(k_i)|}{|\Lambda'_i|} \rho(\delta^x_{\mathbf{t}+K(k_i)}, \mu).$$

Note that in the proof of inequality (4.25) we have used only that $\mathbf{t}_i \in L_i$, $\Pi_i \in \mathrm{Par}(c)$, and $\mathbf{t}_i + \Pi_i \subset K(r_{i+1})$. Hence this inequality will be still valid if we replace therein $\mathbf{t}_i$ by any $\mathbf{t} \in \Lambda_i$ and Π_i by $K(k_i)$. This enables us to estimate $\rho(\delta^x_{\mathbf{t}+K(k_i)}, \mu)$ and obtain

$$\rho(\delta^x_{\Lambda_i'}, \mu) \leqslant \sum_{\mathbf{t} \in \Lambda_i} \frac{|\mathbf{t}+K(k_i)|}{|\Lambda_i'|} \left(\frac{2d}{k_i} + \eta_i + \rho(\mu_i, \mu) \right) = 2d/k_i + \eta_i + \rho(\mu_i, \mu).$$

Now (4.13), (4.3), and (4.6) together lead to (4.27).

To prove (4.28) we note that due to (4.10)

$$\mu_i(C_{\Lambda'}(x)) = \mu_i \left(\bigcap_{\mathbf{t} \in \Lambda_i} C_{\mathbf{t}+K(k_i)}(x) \right) \leqslant \exp(u^{(i)}_{k_i} \cdot |\Lambda_i'|) \prod_{\mathbf{t} \in \Lambda_i} \mu_i(C_{\mathbf{t}+K(k_i)}(x)).$$

Then, using the points $y = y_{x,i,\mathbf{t}}$ introduced in the proof of Lemma 4.2 and the τ-invariance of μ_i, we obtain

$$\begin{aligned} -\frac{\ln \mu_i(C_n(x))}{|\Lambda_i'|} &\geqslant \frac{|K(k_i)|}{|\Lambda_i'|} \sum_{\mathbf{t} \in \Lambda_i} \left[-\frac{\ln \mu_i(C_{\mathbf{t}+K(k_i)}(y))}{|K(k_i)|} \right] - u^{(i)}_{k_i} \\ &= \frac{|K(k_i)|}{|\Lambda_i'|} \sum_{\mathbf{t} \in \Lambda_i} \left[-\frac{\ln \mu_i(C_{K(k_i)}(\tau_{\mathbf{t}} y))}{|K(k_i)|} \right] - u^{(i)}_{k_i}. \end{aligned}$$

It remains to recall that $u^{(i)}_{k_i} \leqslant \eta_i$ (see the definition of k_i) and to employ (4.9). ▮

As before, for straightforward applications, we state the following corollary.

Corollary 4.2. If $i(n)$ is as in Corollary 4.2 and $\Lambda'_{i(n)}$ is defined in (4.26), then for every $x \in \tilde{M}$

$$\lim_{n \to \infty} \rho(\delta^x_{\Lambda'_{i(n)}}, \mu) = 0, \qquad \liminf_{n \to \infty} \left[-\frac{\ln \mu_{i(n)}(C_{\Lambda'_{i(n)}}(x))}{|\Lambda'_{i(n)}|} \right] \geqslant h(\mu).$$

4. The Inclusion $\tilde{M} \subset X_\mu$. For every n there is a unique $i = i(n)$ such that $r_i \leqslant \underline{t}_n < r_{i+1}$. It is evident that $i(n) \to \infty$ together with n. Since $\bar{t}_n / \underline{t}_n \leqslant c$ and $r_{i+1}/r_i \to \infty$ as $i \to \infty$ (see (4.13)), if n is large enough there are only three possibilities: (A) $r_i \leqslant \underline{t}_n \leqslant \bar{t}_n < (l_i + 1)\, r_i$, (B) $(l_i+1)\, r_i \leqslant \bar{t}_n < r_{i+1}$, (C) $r_i \leqslant \underline{t}_n < r_{i+1} \leqslant \bar{t}_n < r_{i+2}$.

We can decompose the sequence $\{T_n\}$ into three subsequences each corresponding to those n that satisfy conditions (A), (B), and (C), respectively. It suffices to prove that $\delta^x_{T_n} \to \mu$ along each of these subsequences

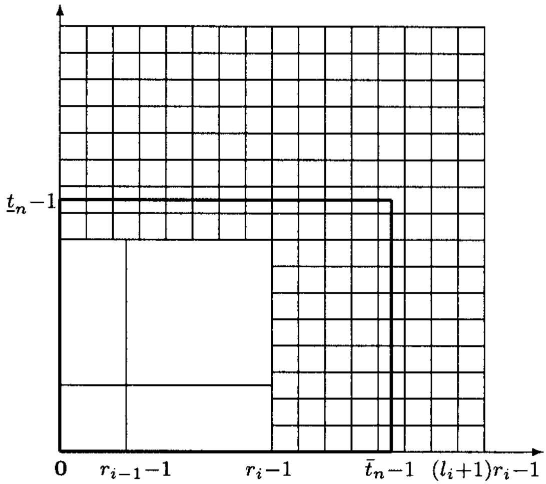

Fig. 1. Location of T_n (type A).

whenever $x \in \tilde{M}$. Therefore, we can assume without loss of generality that one of conditions (A), (B), (C) is satisfied for all n, and thereby consider the sequences of the three types separately.

Sequences of Type A (See Fig. 1). Let $i = i(n)$ and

$$L_{n,i} = \{\mathbf{t} \in L_i : \mathbf{t} + K(k_i) \subset T_n\}, \qquad L'_{n,i} = \bigcup_{\mathbf{t} \in L_{n,i}} (\mathbf{t} + K(k_i)). \tag{4.29}$$

We represent T_n in the form $T_n = T_n^{(1)} \sqcup T_n^{(2)} \sqcup T_n^{(3)}$, where

$$T_n^{(1)} = \mathbf{r}_{i-1} + K(r_i - r_{i-1}), \qquad T_n^{(2)} = L'_{n,i}, \qquad T_n^{(3)} = T_n \setminus (T_n^{(1)} \cup T_n^{(2)}).$$

From (4.21), (4.2) it follows that

$$\begin{aligned} \rho(\delta^x_{T_n}, \mu) \leqslant{} & \frac{|\mathbf{r}_{i-1} + K(r_i - r_{i-1})|}{|T_n|} \cdot \rho(\delta^x_{\mathbf{r}_{i-1}+K(r_i-r_{i-1})}, \mu) \\ & + \frac{|L'_{n,i}|}{|T_n|} \cdot \rho(\delta^x_{L'_{n,i}}, \mu) + \frac{|T_n^{(3)}|}{|T_n|} \cdot \rho(\delta^x_{T_n^{(3)}}, \mu), \quad x \in \tilde{M}. \end{aligned} \tag{4.30}$$

Note that all factors on the right side of (4.30) are bounded. We can apply Corollary 4.1 with $\mathbf{t}_{i(n)} = \mathbf{r}_{i(n)-1}$, $\Pi_{i(n)} = K(r_{i(n)} - r_{i(n)-1})$ to make sure that the first term of this sum tends to 0 as $i \to \infty$, and Corollary 4.2 with $\Lambda'_{i(n)} = L'_{n,\, i(n)}$ to prove the same for the second term (note that, for a sequence of type (A), the relations $\mathbf{t} \in L_{i(n)}$, $\mathbf{t} + K(k_{i(n)}) \subset T_n$ imply $\mathbf{t} + K(k_{i(n)}) \subset D_{i(n)}$). The third term tends to 0 because

$$\frac{|T_n^{(3)}|}{|T_n|} = \frac{|K(r_i)| - |K(r_i - r_{i-1})|}{|T_n|} + \frac{|T_n| - |K(r_i)| - |L'_{n,\,i}|}{|T_n|}$$
$$\leqslant \frac{r_i^d - (r_i - r_{i-1})^d}{r_i^d} + \frac{4d(\bar{t}_n)^{d-1}\, k_i}{\bar{t}_n \underline{t}_n^{d-1}} \leqslant \frac{d r_{i-1}}{r_i} + \frac{4dc^{d-1} k_i}{r_i} \leqslant d\eta_i + 4dc^{d-1}\eta_i$$

(the last inequality follows from (4.13) and (4.12)). Thus, for a sequence $\{T_n\}$ of type A, $\rho(\delta^x_{T_n}, \mu) \to 0$ as $n \to \infty$.

Sequences of Type B (See Fig. 2). We first remind that $T_n = \Pi(\mathbf{t}_n)$. Using the fact that $\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i) \subset T_n$ and taking into account (4.2), (4.5), we obtain

$$\rho(\delta^x_{T_n}, \mu) \leqslant \frac{|\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i)|}{|T_n|}\, \rho(\delta^x_{\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i)}, \mu)$$
$$+ \frac{|T_n \setminus (\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i))|}{|T_n|}\, \rho(\delta^x_{T_n \setminus (\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i))}, \mu), \quad x \in \tilde{M}. \qquad (4.31)$$

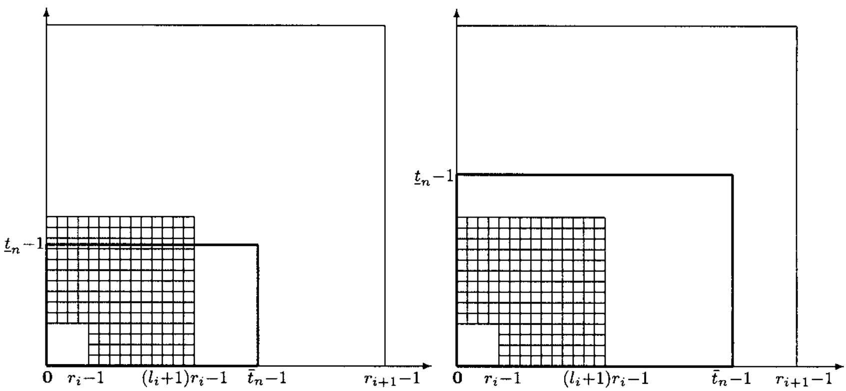

Fig. 2. Location of T_n (type B).

If n (and hence i) is large enough then

$$\frac{\bar{t}_n - r_i}{\underline{t}_n - r_i} = \frac{\bar{t}_n(1 - r_i/\bar{t}_n)}{\underline{t}_n(1 - r_i/\underline{t}_n)} \leqslant \frac{c(1 - r_i/\bar{t}_n)}{1 - r_i/\underline{t}_n}$$

$$\leqslant \frac{c}{1 - r_i/\underline{t}_n} \leqslant \frac{c}{1 - cr_i/\bar{t}_n} \leqslant \frac{c}{1 - c(l_i + 1)} \leqslant 2c,$$

(see conditions (A) and (4.3)) so that $\Pi(\mathbf{t}_n - \mathbf{r}_i) \in \mathrm{Par}(2c)$. Now Corollary 4.1 shows that the first term on the right side of (4.31) tends to 0 as $n \to \infty$. Furthermore, by (4.20)

$$\frac{|T_n| - |\Pi(\mathbf{t}_n - \mathbf{r}_i)|}{|T_n|} \leqslant \frac{dr_i}{\underline{t}_n} \leqslant \frac{cdr_i}{\bar{t}_n} \leqslant \frac{cdr_i}{r_i(l_i + 1)} = \frac{cd}{l_i + 1}, \tag{4.32}$$

where $l_i \to \infty$ (see (4.3)). Therefore $\rho(\delta^x_{T_n}, \mu) \to 0$ for sequences of type B.

Sequences of Type C (See Fig. 3). For such a sequence we have

$$\rho(\delta^x_{T_n}, \mu) \leqslant \frac{|T_n \cap K(r_{i+1})|}{|T_n|} \cdot \rho(\delta^x_{T_n \cap K(r_{i+1})}, \mu)$$
$$+ \frac{|T_n \cap (K(r_{i+2}) \setminus K(r_{i+1}))|}{|T_n|} \cdot \rho(\delta^x_{T_n \cap (K(r_{i+2}) \setminus K(r_{i+1}))}, \mu), \qquad x \in \hat{M}. \tag{4.33}$$

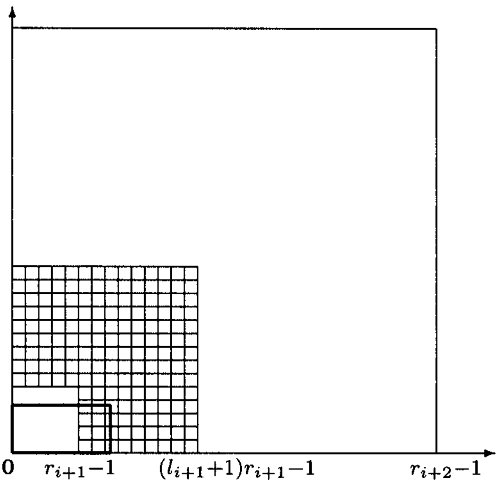

Fig. 3. Location of T_n (type C).

The set $T_n \cap K(r_{i+1})$ is clearly a parallelepiped in $\mathrm{Par}(c)$, and $\{T_n \cap K(r_{i+1})\}$ is a sequence of type B (recall that $i = i(n)$). Hence, as has been already proved, the first term on the right side of (4.31) tends to 0 as $n \to \infty$. At the same time our assumption $\bar{t}_n \leqslant c\underline{t}_n \leqslant cr_{i+1}$ combined with (4.3) implies that for n large enough

$$T_n \cap (K(r_{i+2}) \setminus K(r_{i+1})) \subset D_{i+1}$$

(see (4.14)), and we can prove that the second term tends to 0 as we did when dealing with sequences of type A.

Our final conclusion is: if $x \in \tilde{M}$ then $\rho(\delta^x_{T_n}, \mu) \to 0$ as $n \to \infty$ for a sequence $\{T_n\}$ of any type and hence for any sequence in $\mathrm{Par}(c)$. It means that $\tilde{M} \in X_\mu$ and, since $\tilde{\mu}(\tilde{M}) = 1$, we see that $\tilde{\mu}(X_\mu) = 1$.

5. The behavior of $\ln \tilde{\mu}(C_n(x))$. As before, it suffices to prove (3.3) for sequences of types A – C separately. If $\{T_n\}$ is of type A we have, for every $n \geqslant 1$ and $x \in \tilde{M}$,

$$\tilde{\mu}(C_n(x)) = \tilde{\mu}(C_{T_n \cap K(r_i)}(x))\ \mu_i(C_{T_n \cap (K(r_{i+1}) \setminus K(r_i))}(x)) \qquad (i = i(n))$$

(see the definition of $\tilde{\mu}$), where

$$\tilde{\mu}(C_{T_n \cap K(r_i)}(x)) \leqslant \tilde{\mu}(C_{\mathbf{r}_{i-1}+K(r_i - r_{i-1})}(x)) = \mu_{i-1}(C_{\mathbf{r}_{i-1}+K(r_i - r_{i-1})}(x)),$$

and

$$\mu_i(C_{T_n \cap (K(r_{i+1}) \setminus K(r_i))}(x)) \leqslant \mu_i(C_{\Lambda'_{n,i}}(x)),$$

where $\Lambda'_{n,i} = \bigcup_{\mathbf{t} \in L_{n,i}} (\mathbf{t} + K(k_i))$ (see (4.29)). Hence

$$-\frac{\ln \tilde{\mu}(C_n(x))}{|T_n|} \geqslant \frac{|K(r_i - r_{i-1})|}{|T_n|}\left[-\frac{\ln \mu_{i-1}(C_{\mathbf{r}_{i-1}+K(r_i - r_{i-1})}(x))}{|K(r_i - r_{i-1})|}\right] + \frac{|\Lambda'_{n,i}|}{|T_n|}\left[-\frac{\ln \mu_i(C_{\Lambda'_{n,i}}(x))}{|\Lambda'_{n,i}|}\right].$$

Now we obtain (3.3) from Corollaries 4.1 and 4.2.

For a sequence of type B we have

$$\tilde{\mu}(C_n(x)) \leqslant \tilde{\mu}(C_{\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i)}(x)) = \mu_i(C_{\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i)}(x)), \qquad x \in \tilde{M},$$

so that

$$-\frac{\ln \tilde{\mu}(C_n(x))}{|T_n|} \geqslant \frac{|\Pi(\mathbf{t}_n - \mathbf{r}_i)|}{|T_n|}\left[-\frac{\ln \mu_i(C_{\mathbf{r}_i + \Pi(\mathbf{t}_n - \mathbf{r}_i)}(x))}{|\Pi(\mathbf{t}_n - \mathbf{r}_i)|}\right], \qquad x \in \tilde{M}.$$

Due to (4.32) $|\Pi(\mathbf{t}_n-\mathbf{r}_i)|/|T_n|\to 1$ as $n\to\infty$, and we obtain (3.3) by applying Corollary 4.1.

At last, for a sequence of type C the sets $T_n\cap K(r_{i+1})$, where $i=i(n)$, form a sequence of type B. Therefore, as we have already seen,

$$\liminf_{n\to\infty}\left[-\frac{\ln\tilde{\mu}(C_{T_n\cap K(r_{i(n)+1})}(x))}{|T_n\cap K(r_{i(n)+1})|}\right]\geqslant h(\mu).$$

When dealing with sequences of type A we proved that

$$\liminf_{n\to\infty}\left[-\frac{\ln\mu_{i+1}(C_{T_n\cap(K(r_{i+2})\setminus K(r_{i+1}))}(x))}{|T_n\cap(K(r_{i+2})\setminus K(r_{i+1})|}\right]\geqslant h(\mu)$$

(here we have replaced i by $i+1$). It remains to observe that

$$\begin{aligned}
&-\frac{\ln\tilde{\mu}(C_n(x))}{|T_n|}\\
&\quad=\frac{|T_n\cap K(r_{i+1})|}{|T_n|}\left[-\frac{\ln\mu_i(C_{T_n\cap K(r_{i+1})}(x))}{|T_n\cap K(r_{i+1})|}\right]\\
&\qquad+\frac{|T_n\cap(K(r_{i+2})\setminus K(r_{i+1}))|}{|T_n|}\left[-\frac{\ln\mu_{i+1}(C_{T_n\cap(K(r_{i+2})\setminus K(r_{i+1}))}(x))}{|T_n\cap(K(r_{i+2})\setminus K(r_{i+1}))|}\right].
\end{aligned}$$

Thus Lemma 3.1 is proved.

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REFERENCES

1. H. Cajar, *Billingsley Dimension in Probability Spaces*, Lect. Notes in Math., Vol. 892 (Springer-Verlag, Berlin, 1981).
2. K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications* (Cambridge University Press, Cambridge, 1990).
3. H. Föllmer, On entropy and information gain in random fields, *Z. Wahrsch. verw. Geb.* **26**:207–217 (1973).
4. H.-O. Georgii, *Gibbs Measures and Phase Transitions* (de Gruyter, 1988).
5. B. Gurevich and A. Tempelman, Hausdorff dimension and thermodynamic formalism, *Russian Math. Surveys* **54**:171–172 (1999).
6. B. M. Gurevich and A. A. Tempelman, Hausdorff dimension and pressure in the DLR thermodynamic formalism, in *On Dobrushin's Way. From Probability Theory to Statistical Physics*, (Amer. Math. Soc., Providence, RI, 2000), pp. 91–107.

7. B. M. Gurevich and A. A. Tempelman, Hausdorff dimension of the set of generic points for Gibbs measures (Russian), *Funct. Anal. Appl.* **36**, no. 3 (2002).
8. P. Moran, Additive functions of intervals and Hausdorff dimension, *Proc. Camb. Phil. Society*, **42**:15–23 (1946).
9. E. Olivier, Dimension de Billingsley d'ensembles saturés, *C.R. Acad. Sci. Paris* **328**:13–16 (1999).
10. Ya. Pesin and H. Weiss, On the dimension of deterministic and random Cantor-like sets, symbolic dynamics, and Eckmann–Ruelle conjecture, *Comm. Math. Phys.* **182**:105–153 (1996).
11. D. Ruelle, *Thermodynamic Formalism* (Addison–Wesley, Reading, MA, 1978).
12. A. A. Tempelman, Specific characteristics and variational principle for homogeneous random fields, *Doklady Akademii Nauk SSSR*, **254**:297–302 (1980), (Russian); English translation in *Soviet Math. Doklady*, **22**:363–369 (1980).
13. A. A. Tempelman, Specific characteristics and variational principle for homogeneous random fields, *Z. Wahrsch. Verw. Gebiete*, **65**:341–365 (1984).
14. A. A. Tempelman, *Ergodic Theorems for Group Actions* (Kluwer, 1992).
15. A. A. Tempelman, Dimension of Random Fractals in Metric Spaces, *Teor. Veroyatnost. i Primenen* (Russian); **44**:589–616 (1999). English translation in *Theory Probab. Appl.*, **44**:537–557 (2000).